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MODULE - 04

Numerical Solution of ordinary differential equations of first order and first degree

Numerical methods for initial Value problems

Consider a differential equation of first order and first degree in the form
$\frac{dy}{dx} = f(x,y)$ with the initial condition $y(x_0)=y_0$

i.e $y=y_0$ at $x=x_0$

This problem of finding y is called an initial Value problem.

Taylor's Series method

Consider the initial value problem
$\frac{dy}{dx} = f(x,y)$ and $y(x_0)=y_0$

The solution $y(x)$ is approximated to a power series in $x-x_0$ using Taylor's Series. Then we can find the value of y for various values of x in the neighbourhood of x_0.

we have Taylor's Series expansion $y(x)$ about the point x_0 in the form:

$$y(x) = y(x_0) + (x-x_0)y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0) + \ldots$$

Here $y'(x_0), y''(x_0) \ldots$ denote the value of the derivatives $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots$ at x_0 which can be found by making use of the data

Problem

① (a) Use Taylor's Series method to find y at $x = 0.1, 0.2, 0.3$ considering terms upto the third degree given that $\frac{dy}{dx} = x^2 + y^2$ and $y(0) = 1$

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Sol^n:

Taylor's Series expansion of $y(x)$ is given by

$$y(x) = y(x_0) + (x - x_0) y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots$$

By data

$y(x_0) = y_0 \Rightarrow x_0 = 0 \quad y_0 = 1$ and $y' = x^2 + y^2$

$\Rightarrow y(0) = 1$

$$\therefore y(x) = y(0) + (x-0) y'(0) + \frac{(x-0)^2}{2} y''(0) + \frac{(x-0)^3}{6} y'''(0)$$

$$y(x) = y(0) + x y'(0) + \frac{x^2}{2} y''(0) + \frac{x^3}{6} y'''(0) \quad \text{——} ①$$

we need to compute $y'(0), y''(0), y'''(0)$

Consider $y' = x^2 + y^2$ —— ②

we initial value $y'(0) = 0^2 + 1$

$$\boxed{y'(0) = 1}$$

Differentiating ② w.r.to x we have

$$y'' = 2x + 2yy' \quad \text{——} ③$$

$$y''(0) = 2(0) + 2y(0)y'(0)$$

$$= 0 + 2(1)(1)$$

$$\boxed{y''(0) = 2}$$

Differentiating (3) w. r. to x

$y''' = 2 + 2[yy'' + y'.y']$

$y''' = 2 + 2[yy'' + (y')^2]$

$y'''(0) = 2 + 2[y(0)y''(0) + [y'(0)]^2]$

$y'''(0) = 2 + 2[(1)(2) + 1^2]$

$y'''(0) = 2 + 2[2+1]$

$\boxed{y'''(0) = 8}$

$y'''(x_0) + \dots$

Substituting these values in (1) we have

$y(x) = 1 + x(1) + \frac{x^2}{2}(2) + \frac{x^3}{6}(8)$

$y(x) = 1 + x + x^2 + 4\frac{x^3}{3}$ — (*)

This is called a Taylor's Series approximatio upto third degree and we need to put $x = 0.1, 0.2, 0.3$ in (*) we have

$y(0.1) = 1 + 0.1 + (0.1)^2 + \frac{4(0.1)^3}{3} = 1.1113$ //

$y(0.2) = 1 + 0.2 + (0.2)^2 + \frac{4(0.2)^3}{3} = 1.2506$

$y(0.3) = 1 + 0.3 + (0.3)^2 + \frac{4(0.3)^2}{3} = 1.426$

(2) Find y at $x = 1.02$ correct to five decimal places given $dy = (xy - 1)dx$ and $y = 2$ at $x = 1$ applying Taylor's Series method.

Soln:

Taylor's Series expansion is given by

$$y(x) = y(x_0) + (x-x_0)y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0)$$

By data

$y(x_0) = y_0,\ y(1) = 2 \Rightarrow x_0 = 1 \quad y_0 = 2$ and $y' = \frac{dy}{dx} = xy - 1$

$$y(x) = y(1) + (x-1)y'(1) + \frac{(x-1)^2}{2}y''(1) + \frac{(x-1)^3}{6}y'''(1) \quad \text{—— (1)}$$

Consider $y' = xy - 1$ —— (2)

$y'(1) = (1)y(1) - 1$

use initial value

$= (1)(2) - 1$

$\boxed{y'(1) = 1}$

Differentiate (1) w.r. to x

$y'' = xy' + y(1)$

$y'' = xy' + y$ —— (3)

use initial value and $y'(1)$ value

$y''(1) = (1)y'(1) + y(1)$

$y''(1) = (1)(1) + 2$

$\boxed{y''(1) = 3}$

Differentiate (3) w.r. to x

$y''' = xy'' + y'(1) + y'$

$y''' = xy'' + 2y'$

$y'''(1) = (1)\,y''(1) + 2y'(1)$

$y'''(1) = (1)(3) + 2(1)$

$\boxed{y'''(1) = 5}$

we need to find $y(1.02)$

put $x = 1.02$ in ①

$$y(1.02) = 2 + (1.02-1)(1) + \frac{(1.02-1)^2}{2}(3) + \frac{(1.02-1)^3}{6} \times 5$$

$$= 2 + 0.02 + \frac{(0.02)^2}{2} \times 3 + \frac{(0.02)^3}{6} \times 5$$

$$= 2 + 0.02 + 0.0006 + 0.000006$$

$$y(1.02) = 2.020606$$

③ From Taylor's Series method, find $y(0.1)$ Considering upto fourth degree term if $y(x)$ satisfies the equation $\frac{dy}{dx} = x - y^2$, $y(0) = 1$

Solⁿ: Taylor's Series expansion is given by

$$y(x) = y(x_0) + (x-x_0)y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0) + \frac{(x-x_0)^4}{4!}y^{4}(x_0) + \cdots$$

By data $x_0 = 0$, $y_0 = 1$ i.e $y(0) = 1$

$$y' = x - y^2$$

$$y(x) = y(0) + (x-0)y'(0) + \frac{(x-0)^2}{2!}y''(0) + \frac{(x-0)^3}{3!}y'''(0) + \frac{(x-0)^4}{4!}y^4(0)$$

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2}y''(0) + \frac{x^3}{6}y'''(0) + \frac{x^4}{24}y^4(0) \quad \text{— (1)}$$

Consider $y' = x - y^2$ — (2), $y'(0) = 0 - [y(0)]^2$

$y'(0) = 0 - 1^2 = -1$

$\therefore \boxed{y'(0) = -1}$

Differentiate (2) w. r. to x

$y'' = 1 - 2yy'$ — (3); $y''(0) = 1 - 2y(0)y'(0)$

$y''(0) = 1 - 2(1)(-1)$

$y''(0) = 1 + 2$

$\boxed{y''(0) = 3}$

Diff. (3) w. r. to x

$y''' = 0 - 2[yy'' + y'y']$

$y''' = -2[yy'' + (y')^2]$ — (4)

$y'''(0) = -2[y(0)y''(0) + \{y'(0)\}^2]$

$= -2[(1)(3) + (-1)^2]$

$y'''(0) = -2[3+1]$

$\frac{-0)^4}{4!} y^4(0)$

$$\boxed{y'''(0) = -8}$$

Diff. (4) w.r. to x

$-$ (1)

$$y^{(4)} = -2\left[y y''' + y''y' + 2y'y''\right]$$
$$= -2\left[y y''' + 3y'y''\right]$$
$$y^{(4)}(0) = -2\left[y(0)\, y'''(0) + 3\, y'(0)\, y''(0)\right]$$

$$= -2\left[(1)(-8) + 3(-1)(3)\right]$$
$$= -2\left[-8 - 9\right]$$
$$= -2 \times -17$$

$$\boxed{y^{(4)}(0) = 34}$$

Subytitute all these valuy in (1)

$$y(x) = 1 + x(-1) + \frac{x^2}{2}(3) + \frac{x^3}{6}(-8) + \frac{x^4}{\cancel{24}_{12}} \times \cancel{34}^{17}$$

$$y(x) = 1 - x + \frac{3x^2}{2} - \frac{8x^3}{6} + \frac{17x^4}{12} \quad \text{—— } (*)$$

we need to find $y(0.1)$

put $x = 0.1$ in $(*)$

$$y(0.1) = 1 - (0.1) + \frac{3(0.1)^2}{2} - \frac{8(0.1)^3}{6} + \frac{17(0.1)^4}{12}$$

$$= 1 - 0.1 + 0.015 - 0.00136 + 0.00014$$

$$\boxed{y(0.1) = 0.91384}$$

(H) Use Taylor's Series method to find $y(4.1)$

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Solⁿ: Taylor's Series expansion is given by

$$y(x) = y(x_0) + (x-x_0)y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0) + \cdots$$

By data $y(x_0) = y_0$

$y(4) = 4$

i.e $x_0 = 4$ $y_0 = 4$, $y' = \frac{1}{x^2+y}$

$$y(x) = y(4) + (x-4)y'(4) + \frac{(x-4)^2}{2}y''(4) + \frac{(x-4)^3}{6}y'''(4) - $$

Consider $y' = \frac{1}{x^2+y}$

$y'(x^2+y) = 1$ — ②

Substituting the initial values

$y'(4)[4^2 + y(4)] = 1$

$y'(4)[16+4] = 1 \Rightarrow y'(4)[20] = 1$

$\Rightarrow y'(4) = \frac{1}{20} = 0.05 \Rightarrow \boxed{y'(4) = 0.05}$

Diff. ② w.r. to x

$y'[2x + y'] + (x^2+y)y'' = 0$ — ③

Substituting initial values and value of $y'(4)$

$y'(4)[2(4) + y'(4)] + [4^2 + y(4)]y''(4) = 0$

$0.05[8 + 0.05] + [16+4]y''(4) = 0$

$0.4025 + 20y''(4) = 0$

$20y''(4) = -0.4025$

$y''(4) = \frac{-0.4025}{20}$

$\boxed{y''(4) = -0.020125}$

Diff. (5) w.r. to x — 3rd degree term can be neglected ∵ values of derivatives are very small

$y'[2 + y''] + [2x + y']y'' + [x^2 + y]y''' + y''[2x+y'] = 0$ —(1)

$2y' + y'y'' + 2xy'' + y'y'' + x^2y''' + yy''' + 2xy'' + y'y'' = 0$

$4xy'' + 3y'y'' + 2y' + x^2y''' + yy''' = 0$

$4(4)y''(4) + 3y'(4)y''(4) + 2y'(4) + (4)^2y'''(4) + y(4)y'''(4) = 0$

$16(-0.020125) + 3(0.05)(-0.020125) + 2(0.05) + 16y'''(4) + (4)y'''(4) = 0$

$-0.22502 + 20y'''(4) = 0$

$20y'''(4) = 0.22502$

$\boxed{y'''(4) = +0.01125}$

put all these values in (1)

$y(x) = 4 + (x-4)(0.05) + \frac{(x-4)^2}{2}(-0.020125) + \frac{(x-4)^3}{6}(-0.01125)$ — (*)

we need to find $y(4.1)$

put $x = 4.1$ in (*)

$y(4.1) = 4 + (4.1-4)(0.05) + \frac{(4.1-4)^2}{2}(-0.020125) + \frac{(4.1-4)^3}{6}(-0.01125)$

$y(4.1) = 4 + 0.005 - 0.0001 - 0.0000018$

$\boxed{y(4.1) = 4.00489}$

(5) (a) Employ Taylor's Series method to find y at $x = 0.1$ and 0.2 correct to four places of decimal in step size of 0.1 given the linear differential equation $\frac{dy}{dx} - 2y = 3e^x$ whose solution passes through the origin. Also find $y(0.1)$ and $y(0.2)$ by analytical method

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(b) Compute $y(0.1)$

Sol^n:

$$y(x) = y(x_0) + (x - x_0)y'(x_0) + \frac{(x - x_0)^2}{2!}y''(x_0) + \frac{(x - x_0)^3}{3!}y'''(x_0) + \ldots$$

origin $(0,0)$

By data $y(0) = 0$

i.e $x_0 = 0$ $y_0 = 0$, $y' - 2y = 3e^x$

$y' = 3e^x + 2y$

$$y(x) = y(0) + (x-0)y'(0) + \frac{(x-0)^2}{2}y''(0) + \frac{(x-0)^3}{6}y'''(0)$$

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2}y''(0) + \frac{x^3}{6}y'''(0) \quad \text{— (1)}$$

Step (1): we need to find $y(0.1)$

Consider $y' = 3e^x + 2y$ —— (2)

We initial conditions

$y'(0) = 3e^0 + 2y(0)$

$y'(0) = 3(1) + 2(0)$

$\boxed{y'(0) = 3}$

Diff. ② or y' w. r. to x

$y'' = 3e^x + 2y'$ ——— ③

use initial condition & value of $y'(0)$

$y''(0) = 3e^0 + 2y'(0)$

$y''(0) = 3(1) + 2(3)$

$y''(0) = 9$

Diff. ③ w. r. to x

$y''' = 3e^x + 2y''$

$y'''(0) = 3e^0 + 2y''(0)$

$y'''(0) = 3(1) + 2(9)$

$y'''(0) = 21$

we need to find $y(0\cdot1)$

Now eqn ① becomes

$$y(x) = 0 + 3x + \frac{x^2}{2}(9) + \frac{x^3}{6}(21) \quad — \ (*)$$

put $x = 0\cdot1$ in $(*)$

$$y(0\cdot1) = 0 + 3(0\cdot1) + \frac{(0\cdot1)^2(9)}{2} + \frac{(0\cdot1)^3(21)}{6}$$

$$y(0\cdot1) = 0\cdot3 + 0.045 + 0.0035$$

$$y(0\cdot1) = 0\cdot3485$$

Step ②: we shall find $y(0\cdot2)$

$y(0\cdot1) = 0\cdot3485$

i.e $x_0 = 0\cdot1$ $y_0 = 0\cdot3485$

Taylors Series expansion is given by

①

$$y(x) = y(x_0) + (x-x_0)y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0) + \dots$$

$$y(x) = y(0\cdot1) + (x-0\cdot1)y'(0\cdot1) + \frac{(x-0\cdot1)^2}{2}y''(0\cdot1) + \frac{(x-0\cdot1)^3}{6}y'''$$ — (4)

Consider $y' = 3e^x + 2y$ — (5)

$y'(0\cdot1) = 3e^{0\cdot1} + 2y(0\cdot1) \Rightarrow y'(0\cdot1) = 3e^{0\cdot1} + 2(0.3485)$

$y'(0\cdot1) = 4\cdot0125$ //

Diff (5) w.r. to x

$y'' = 3e^x + 2y'$ — (6)

$y''(0\cdot1) = 3e^{0\cdot1} + 2y'(0\cdot1) \Rightarrow y''(0\cdot1) = 3e^{0\cdot1} + 2(4\cdot0125)$

$y''(0\cdot1) = 11\cdot3405$ //

Diff (6) w.r. to x

$y''' = 3e^x + 2y''$

$y'''(0\cdot1) = 3e^{0\cdot1} + 2y''(0\cdot1) \Rightarrow y'''(0\cdot1) = 3e^{0\cdot1} + 2(11\cdot3405)$

$y'''(0\cdot1) = 25\cdot9965$ //

put all these values in (4)

$$y(x) = 0\cdot3485 + (x-0\cdot1)(4\cdot0125) + \frac{(x-0\cdot1)^2}{2}(11\cdot3405) + \frac{(x-0\cdot1)^3}{6}25\cdot9965$$

we need to find $y(0\cdot2)$, So put $x = 0\cdot2$

$$y(0\cdot2) = 0\cdot3485 + (0\cdot2-0\cdot1)(4\cdot0125) + \frac{(0\cdot2-0\cdot1)^2}{2}(11\cdot3405) + \frac{(0\cdot2-0\cdot1)^3}{6} \times 25\cdot9965$$

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$$y(0.2) = 0.3485 + (0.1)(4.0125) + \frac{(0.1)^2}{2}(11.3405)$$

$$+ \frac{(0.1)^3}{6}(25.9965)$$

$$= 0.3485 + 0.40125 + 0.0567025 + 0.004332$$

$$= 0.81078$$

$$y(0.2) = 0.8108$$

Thus

Solution by analytic method

$\frac{dy}{dx} - 2y = 3e^x$ is of the form $\frac{dy}{dx} + Py = Q$

where $P = -2$, $Q = 3e^x$

Solution : $ye^{\int P dx} = \int Q e^{\int P dx} dx + C$

$$ye^{\int -2dx} = \int 3e^x e^{\int -2dx} dx + C$$

$$ye^{-2x} = \int 3e^x e^{-2x} dx + C$$

$$ye^{-2x} = \int 3e^{-x} dx + C$$

$$ye^{-2x} = 3\frac{e^{-x}}{-1} + C$$

$$ye^{-2x} = -3e^{-x} + C$$

$$y = \frac{-3e^{-x}}{e^{-2x}} + \frac{C}{e^{-2x}}$$

$$y = -3e^{-x} \cdot e^{2x} + ce^{2x}$$

$$y = -3e^{-x+2x} + ce^{2x}$$

$y = -3e^x + ce^{2x}$ is general solution

apply initial condition $y(0) = 0$ in the above equation

$$y(0) = -3e^0 + ce^0$$

$$0 = -3 + C$$

$$C = 3$$

put $C = 3$ in $y = -3e^{x} + Ce^{2x}$

$y = -3e^{x} + 3e^{2x}$

$y = 3(e^{2x} - e^{x})$ is the solution

Let us find $y(0.1)$ and $y(0.2)$ so

put $x = 0.1$ in the above sol^n $y(0.1) = 3(e^{2(0.1)} - e^{0.1})$

$y(0.1) = 0.34869$

$y(0.1) = 0.3487$

put $x = 0.2$ in the above solution

$y(0.2) = 3(e^{2(0.2)} - e^{0.2})$

$= 0.81126$

$y(0.2) = 0.8113$ by analytical method

(6) Using Taylor's Series method, obtain the Values of y at $x = 0.1, 0.2, 0.3$ to four Significant figures if y satisfies the equation $y'' = -xy$ given that $y' = 0.5$ and $y = 1$ when $x = 0$ taking the first five terms of the Taylor's Series expansion

Sol^{n}: Taylor's Series expansion is given by

$$y(x) = y(x_0) + (x-x_0)y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0) +$$

By data $x_0 = 0$, $y_0 = 1$

$y' = 0.5$

$y'' = -xy$

put $x_0 = 0$ in above formula

$$y(x) = y(0) + x\,y'(0) + \frac{x^2}{2}y''(0) + \frac{x^3}{6}y'''(0) \quad \text{— (1)}$$

consider $y'' = -xy$ — (2)

$y''(0) = -(0)(1) = 0 \Rightarrow y''(0) = 0$

eqn (2) D.w.r.to x

$y''' = -[xy' + y(1)]$

$y''' = -xy' - y$ — (3)

$y'''(0) = -(0)y'(0) - y(0)$

$y'''(0) = -(0)(0.5) - 1$

$y'''(0) = -1$

put all these values in (1)

$$y(x) = 1 + x(0.5) + \frac{x^2}{2}(0) + \frac{x^3}{6}(-1) \quad \text{— } (*)$$

Now we need to find $y(0.1)$, $y(0.2)$ & $y(0.3)$

$$y(0.1) = 1 + (0.1)(0.5) + \frac{(0.1)^2}{2}(0) + \frac{(0.1)^3}{6}(-1)$$

$y(0.1) = 1.0498$

put $x = 0.2$ in $(*)$

$$y(0.2) = 1 + (0.2)(0.5) + \frac{(0.2)^2}{2}(0) + \frac{(0.2)^3}{6}(-1)$$

$y(0.2) = 1.0986$

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$$y(0.3) = 1 + (0.3)(0.5) - \frac{(0.3)^2}{2}(0) + \frac{(0.3)^3}{6} \times -1$$

$$= 1 + 0.15 - 0 - 0.0045$$

$$= 1.1455$$

$\therefore y(0.1) = 1.00498,\ y(0.2) = 1.0986 \quad y(0.3) = 1.1455$

⑦ Use Taylor's Series method to solve $y' = x^2 + y$ in the range $0 \le x \le 0.2$ by taking stepsize $h = 0.1$ given that $y = 10$ at $x = 0$ initially considering terms upto the fourth degree.

Solⁿ:

In this problem

Since the step size is specified as 0.1, the problem has to be done in two stages we have to first find $y(0.1)$ and use this as the initial Condition to find $y(0.2)$

Taylor's Series expansion is given by

$$y(x) = y(x_0) + (x - x_0)y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0)$$

$$+ \frac{(x-x_0)^4}{4!}y^{4}(x_0) + \ldots \quad (*)$$

I stage: By data

$$y' = x^2 + y, \quad x_0 = 0 \quad y_0 = 10$$

$y'(0) = 0^2 + y(0)$ use initial conditions

$$y'(0) = 0 + 10$$

$$y'(0) = 10$$

Differentiating y' w. r. to x

$y'' = 2x + y'$

$y''(0) = 2(0) + y'(0) \Rightarrow y''(0) = 0 + 10 \Rightarrow y''(0) = 10$ //

Diff. y'' w. r. to x

$y''' = 2 + y''$

$y'''(0) = 2 + y''(0) \Rightarrow y'''(0) = 2 + (10) \Rightarrow y'''(0) = 12$ //

D. y''' w. r. to x

$y^4 = 0 + y'''$

$y^4(0) = 0 + y'''(0) \Rightarrow y^4(0) = 12$ //

Now we have to find y at $x = 0.1$

So put $x = 0.1$ with $x_0 = 0$

$\therefore$ (*) becomes

$$y(0.1) = y(0) + (0.1-0)y'(0) + \frac{(0.1-0)^2}{2}y''(0) + \frac{(0.1-0)^3}{6}y'''(0) + \frac{(0.1-0)^4}{24}y^4(0)$$

$$y(0.1) = 10 + (0.1)(10) + \frac{(0.1)^2}{2}(10) + \frac{(0.1)^3}{6}(12) + \frac{(0.1)^4}{24}(12)$$

$$= 10 + 1 + 0.05 + 0.002 + 0.00005$$

$$y(0.1) = \underline{11.05205} \approx \underline{11.052}$$

II Stage: Now take $x_0 = 0.1$, $y_0 = 11.052$

we have

$y' = x^2 + y$

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We initial conditions

$y'(0.1) = (0.1)^2 + 11.052 = 11.062 \Rightarrow y'(0.1) = 11.062$ //

Diff y' w. r. to x

$y'' = 2x + y'$;

$y''(0.1) = 2(0.1) + y'(0.1) \Rightarrow y''(0.1) = 0.2 + 11.062$

$y''(0.1) = 11.262$ //

Diff. y'' w. r. to x

$y''' = 2 + y''$

$y'''(0.1) = 2 + y''(0.1) \Rightarrow y'''(0.1) = 2 + 11.262$

$y'''(0.1) = 13.262$ //

Diff y''' w. r. to x

$y^4 = 0 + y'''$

$y^4(0.1) = y'''(0.1) \Rightarrow y^4(0.1) = 13.262$ //

Now we have to find $y(0.2)$

[i.e y at $x = 0.2$]

So put $x = 0.2$ with $x_0 = 0.1$

(*) becomes

$$y(0.2) = y(0.1) + (0.2-0.1)y'(0.1) + \frac{(0.2-0.1)^2}{2}y''(0.1)$$

$$+ \frac{(0.2-0.1)^3}{6}y'''(0.1) + \frac{(0.2-0.1)^4}{4!}y^4(0.1)$$

$$= 11.052 + (0.1)11.062 + \frac{(0.1)^2}{2}(11.262) + \frac{(0.1)^3}{6}(13.262)$$

$$+ \frac{(0.1)^4}{24} \times 13.262$$

$= 11.052 + 1.1062 + 0.05631 + 0.00221 + 0.000055$

$y(0.2) = 12.21677$

Thus $y(0.1) = 11.052$ and $y(0.2) = 12.21677$

(8) Use Taylor's Series method to obtain a power series in $(x-4)$ for the equation $5x\frac{dy}{dx} + y^2 - 2 = 0$, $x_0 = 4$, $y_0 = 1$ and use it to find y at $x = 4.1, 4.2, 4.3$ correct to four decimal places.

Sol^n: Taylor's Series expansion is given by

$$y(x) = y(x_0) + (x-x_0)y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0) + \dots$$

By data $x_0 = 4$, $y_0 = 1$

So

$$y(x) = y(4) + (x-4)y'(4) + \frac{(x-4)^2}{2}y''(4) + \frac{(x-4)^3}{6}y'''(4) \quad \text{--- (1)}$$

Consider $5xy' + y^2 - 2 = 0$ —— (2)

Substitute initial values we obtain

[NOTE: $y' = y'(x)$]

$$5(4)y'(4) + (y(4))^2 - 2 = 0$$

$$20y'(4) + (1)^2 - 2 = 0$$

$$20y'(4) = 1$$

$$y'(4) = 1/20 = 0.05$$

$$y'(4) = 0.05$$

Diff. ② w.r. to x

$$5[xy'' + y' \times 1] + [2yy' - 0] = 0$$

$$5[xy'' + y'] + [2yy'] = 0 \quad \text{—— ③}$$

use initial condition

$$5[(4)y''(4) + y'(4)] + [2y(4)y'(4)] = 0$$

$$5[4y''(4) + 0.05] + [2(1)(0.05)] = 0$$

$$20y''(4) + 0.25 + 0.1 = 0$$

$$20y''(4) + 0.35 = 0$$

$$20y''(4) = -0.35$$

$$y''(4) = -0.0175$$

Since the value of the second derivative itself y small enough we shall approximate up to second degree only

Sub. all value in ①

$$y(x) = 1 + (x-4)(0.05) + \frac{(x-4)^2}{2}(-0.0175) \quad \text{—— (*)}$$

Now we have to find y at $x = 4.1, 4.2, 4.3$

So, put $x = 4.1$ in (*)

$$y(4.1) = 1 + (4.1-4)(0.05) + \frac{(4.1-4)^2}{2}(-0.0175)$$

$$y(4.1) = 1.0049$$

put $x = 4.2$ in (*)

$$y(4.2) = 1 + (4.2-4)(0.05) + \frac{(4.2-4)^2}{2}(-0.0175)$$

$y(4.2) = 1.0097$ //

put $x = 4.3$ in (*)

$y(4.3) = 1 + (4.3-4)(0.05) + \frac{(4.3-4)^2}{2}(-0.0175)$

$y(4.3) = 1.0142$

(OR)

Diff (2) w.r. to x

If you consider derivative upto 3rd degree answer will not change

$5[xy''' + y'' + y''] + 2[yy'' + y'y'] = 0$

$5[xy''' + 2y''] + 2[yy'' + (y')^2] = 0$

$5[(4)y'''(4) + 2y''(4)] + 2[y(4)y''(4) + (y'(4))^2] = 0$

$5[4y'''(4) + 2(-0.0175)] + 2[(1)(-0.0175) + (0.05)^2] = 0$

$20y'''(4) - 0.175 + (-0.035) + 0.005 = 0$

$20y'''(4) - 0.205 = 0$

$y'''(4) = \frac{0.205}{20}$

$y'''(4) = 0.01025$ //

put all these in (1)

$y(x) = 1 + (x-4)(0.05) + \frac{(x-4)^2}{2}(-0.0175) + \frac{(x-4)^3}{6}(0.01025)$

we need to find y at $x = 4.1, 4.2, 4.3$

$y(x) - y(4.1) = 1.0049$

$y(4.2) = 1.0097$

$y(4.3) = 1.0142$

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Modified Euler's Method

Consider the initial value problem

$$\frac{dy}{dx} = f(x, y) \quad ; \quad y(x_0) = y_0$$

we need to find y at $x_1 = x_0 + h$
we first obtain $y(x_1) = y_1$ by applying Euler's formula and this value is regarded as the initial approximation for y_1 usually denoted by $y_1^{(0)}$ also called as predicted value of y_1.

Euler's formula is given by

$$y_1^{(0)} = y_0 + h f(x_0, y_0)$$

Since the accuracy is poor in this formula this value y_1 is successively corrected to the desired degree of accuracy by the following modified Euler's formula where the successive approximations are denoted by $y_1^{(1)}, y_1^{(2)}, y_1^{(3)} \ldots$ etc

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right]$$

Euler's formula and ~~Euler's~~ modified Euler's formula jointly called as Euler's predictor and corrector formula

Problems

① ⓐ Given $\frac{dy}{dx} = 1 + \frac{y}{x}$, $y = 2$ at $x = 1$ find the approximate value of y at $x = 1.4$ by taking step size $h = 0.2$ applying modified Euler's method. Also find the value of y at $x = 1.2$ and 1.4 from the analytical solution of the equation.

Dec 2018

ⓑ Find $y(1.2)$ in two steps.

Sol^n: The problem has to be done in 2 stages.

I Stage:

$x_0 = 1$, $y_0 = 2$ $\quad \frac{dy}{dx} = f(x, y) = 1 + \frac{y}{x}$ $\quad h = 0.2$

$x_1 = x_0 + h = 1 + 0.2 = 1.2 \Rightarrow x_1 = 1.2$

$y(x_1) = y_1 = y(1.2) = ?$

Now

$$f(x_0, y_0) = 1 + \frac{y_0}{x_0} = 1 + \frac{2}{1} = 1 + 2 = 3$$

$f(x_0, y_0) = 3$

we have Euler's formula

$$y_1^{(0)} = y_0 + h f(x_0, y_0)$$
$$= 2 + 0.2\,(3)$$
$$= 2.6$$

we have modified Euler's formula

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$
$$= 2 + \frac{0.2}{2}\left[3 + \left\{1 + \frac{y_1^{(0)}}{x_1}\right\}\right]$$
$$= 2 + 0.1\left[3 + 1 + \frac{2.6}{1.2}\right]$$

$y_1^{(1)} = 2.61666$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$= 2 + \frac{0.2}{2}\left[3 + \left\{1 + \frac{y_1^{(1)}}{x_1}\right\}\right]$$

$$= 2 + 0.1\left[3 + 1 + \frac{2.61666}{1.2}\right]$$

$y_1^{(2)} = 2.61805$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right]$$

$$= 2 + \frac{0.1}{2}\left[3 + \left\{1 + \frac{y_1^{(2)}}{x_1}\right\}\right]$$

$$= 2 + 0.1\left[3 + 1 + \frac{2.61805}{1.2}\right]$$

$= 2.61817$

$$y_1^{(4)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(3)})\right]$$

$$= 2 + \frac{0.2}{2}\left[3 + \left\{1 + \frac{y_1^{(3)}}{x_1}\right\}\right]$$

$$= 2 + 0.1\left[3 + 1 + \frac{2.61817}{1.2}\right]$$

$= 2.61818$

$\therefore \; y(1.2) = 2.61818$

Required solution for (b) is

$y_1^{(2)} = y(1.2) = 2.61805$

II stage: we repeat the process by taking $y(1.2) = 2.61818$ as the initial condition

$x_0 = 1.2 \quad y_0 = 2.61818$

$f(x_0, y_0) = 1 + \frac{y_0}{x_0} = 1 + \frac{2.61818}{1.2} = 3.1818$

$f(x_0, y_0) = 3.1818$, $x_1 = x_0 + h \Rightarrow x_1 = 1.4$, $y(x_1) = y_1 = y(1.4) = ?$

we have Eulers formula

$y_1^{(0)} = y_0 + h f(x_0, y_0)$

$= 2.61818 + (0.2)(3.1818)$

$= 3.25454$

Now we have modified Euler's formula

$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$

$= 2.61818 + \frac{0.2}{2}\left[3.1818 + 1 + \frac{y_1^{(0)}}{x_1}\right]$

$= 2.61818 + 0.1\left[3.1818 + 1 + \frac{3.25454}{1.4}\right]$

$= 3.2688$

$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$

$= 2.61818 + \frac{0.2}{2}\left[3.1818 + 1 + \frac{y_1^{(1)}}{x_1}\right]$

$= 2.61818 + 0.1\left[3.1818 + 1 + \frac{3.2688}{1.4}\right]$

$= 3.2698$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right]$$

$$= 2.61818 + \frac{0.2}{2}\left[3.1818 + \left\{1 + \frac{y_1^{(2)}}{x_1}\right\}\right]$$

$$= 2.61818 + 0.1\left[3.1818 + 1 + \frac{3.2698}{1.4}\right]$$

$$= 2.61818 + 0.1\left[3.1818 + 1 + 2.335571429\right]$$

$$= 3.2699$$

Thus $y(1.4) = 3.2699$

Now, let us find the analytical solution of the equation

$$\frac{dy}{dx} = 1 + \frac{y}{x} \quad \text{or} \quad \frac{dy}{dx} - \frac{y}{x} = 1$$

This is a linear DE of the form $\frac{dy}{dx} + Py = Q$ whose solution is given by

$$y e^{\int p\,dx} = \int Q\, e^{\int p\,dx}\, dx + C$$

here $p = -1/x$ & $Q = 1$

$$e^{\int p.dx} = e^{\int -1/x\,dx} = e^{-\log x} = e^{\log x^{-1}} = x^{-1} = 1/x$$

Solution becomes

$$y \cdot 1/x = \int 1 \cdot 1/x\, dx + C = \int 1/x\, dx + C$$

$$y/x = \log x + C \quad - (*)$$

apply the initial condition

i.e $y = 2$ & $x = 1$ we have

$$2/1 = \log(1) + C \Rightarrow 2 = 0 + C \Rightarrow C = 2$$

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put c value in (*)

$$y/x = \log x + 2$$

$$y = x(\log x + 2)$$

This is the analytical solution of given initial value problem

Now by putting $x = 1.2$ & 1.4 in the above we get $y = 1.2(\log_e(1.2) + 2)$

$$y = 2.61878$$

put $x = 1.4$

$$y = 1.4(\log_e(1.4) + 2)$$

$$y = 3.27106$$

② ⓐ Using modified Euler's method find y at $x = 0.2$ given $\frac{dy}{dx} = 3x + \frac{1}{2}y$ with $y(0) = 1$

DEC 2017 taking $h = 0.1$. Perform three iterations at each step.

Sol^n: We need to find $y(0.2)$ by taking $h = 0.1$. The Problem has to be done in two stages

I Stage: By data $x_0 = 0$, $y_0 = 1$ $h = 0.1$

$$\frac{dy}{dx} = f(x, y) = 3x + \frac{1}{2}y$$

$$f(x_0, y_0) = 3(0) + \frac{1}{2}(1)$$

$$f(x_0, y_0) = \frac{1}{2} = 0.5$$

$$x_1 = x_0 + h = 0 + 0.1$$

$$x_1 = 0.1$$

$$y(x_1) = y_1 = y(0.1) = ?$$

from Euler's formula

$$y_1^{(0)} = y_0 + h f(x_0, y_0)$$

$$= 1 + (0.1)(0.5)$$

$$y_1^{(0)} = \underline{1.05}$$

we have modified Euler's formula

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$= 1 + \frac{0.1}{2}\left[0.5 + 3x_1 + \frac{y_1^{(0)}}{2}\right]$$

$$= 1 + 0.05\left[0.5 + 3(0.1) + \frac{1.05}{2}\right]$$

$$= 1 + 0.05\left[0.5 + 0.3 + 0.525\right]$$

$$y_1^{(1)} = 1.06625$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$= 1 + \frac{0.1}{2}\left[0.5 + 3x_1 + \frac{y_1^{(1)}}{2}\right]$$

$$= 1 + 0.05\left[0.5 + 3(0.1) + \frac{1.06625}{2}\right]$$

$$= 1 + 0.05\left[0.5 + 0.3 + 0.533125\right]$$

$$= 1.06665$$

$$y\ \underline{1.0667}$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$= 1 + 0.05\left[0.5 + 3x_1 + \frac{y_1^{(1)}}{2}\right]$$

$$= 1 + 0.05\left[0.5 + 3(0.1) + \frac{1.0667}{2}\right]$$

$$= 1 + 0.05\left[0.5 + 0.3 + 0.53335\right]$$

$$= 1.06666$$

$$= 1.0667$$

Thus $y(0.1) = 1.0667$

II Stage: Now, let $x_0 = 0.1$, $y_0 = 1.0667$

we have $f(x, y) = 3x + \frac{y}{2}$

$$f(x_0, y_0) = 3(0.1) + \frac{1.0667}{2} = 0.83335$$

$x_1 = x_0 + h = 0.2$; $y_1 = y(x_1) = y(0.2) = ?$

From Euler's formula we obtain

$$y_1^{(0)} = y_0 + hf(x_0, y_0)$$

$$= 1.0667 + 0.1(0.83335)$$

$$= 1.150035$$

$$\approx 1.15$$

From modified Euler's formula

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$= 1.0667 + \frac{0.1}{2}\left[0.83335 + 3x_1 + \frac{y_1^{(0)}}{2}\right]$$

$$= 1.0667 + 0.05\left[0.83335 + 3(0.2) + \frac{1.15}{2}\right]$$

$$y_1^{(1)} = 1.1671$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$= 1.0667 + 0.05\left[0.83335 + 3x_1 + \frac{y_1^{(1)}}{2}\right]$$

$$= 1.0667 + 0.05\left[0.83335 + 3(0.2) + \frac{1.1671}{2}\right]$$

$$y_1^{(2)} = 1.1675$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right]$$

$$= 1.0667 + 0.05\left[0.83335 + 3x_1 + \frac{y_1^{(2)}}{2}\right]$$

$$= 1.0667 + 0.05\left[0.83335 + 3(0.2) + \frac{1.1675}{2}\right]$$

$$y_1^{(3)} = 1.1675$$

Thus $y(0.2) = 1.1675$

③ Using modified Euler's method find $y(0.2)$ correct to four decimal places solving the equation $\frac{dy}{dx} = x - y^2$, $y(0) = 1$ taking $h = 0.1$

June 2018

(4) Using modified Euler's method find $y(20.2)$ and $y(20.4)$ given that $\frac{dy}{dx} = \log_{10}\left(\frac{x}{y}\right)$ with $y(20) = 5$ taking $h = 0.2$

Soln: we shall first find $y(20.2)$ and use this value to find $y(20.4)$

I stage: By data

$x_0 = 20,\ y_0 = 5$ and $h = 0.2$

$f(x,y) = \log_{10}\left(\frac{x}{y}\right)$; $f(x_0, y_0) = \log_{10}(20/5)$

$f(x_0,y_0) = \log_{10}(4) = 0.60205$

$f(x_0,y_0) = 0.6021$

$x_1 = x_0 + h = 20 + 0.2 = 20.2$

$y(x_1) = y_1 = y(20.2) = ?$

From Euler's formula : $y_1^{(0)} = y_0 + hf(x_0, y_0)$

$$y_1^{(0)} = y_0 + hf(x_0, y_0)$$
$$= 5 + 0.2(0.6021)$$
$$= 5.1204$$

By Euler's modified formula

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$= 5 + \frac{0.2}{2}\left[0.6021 + \log_{10}\left(x_1/y_1^{(0)}\right)\right]$$

$$= 5 + 0.1\left[0.6021 + \log_{10}\left(\frac{20.2}{5.1204}\right)\right]$$

$$y_1^{(1)} = 5.1198$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$= 5 + 0.1\left[0.6021 + \log_{10}\left(\frac{x_1}{y_1^{(1)}}\right)\right]$$

$$= 5 + 0.1\left[0.6021 + \log_{10}\left(\frac{20.2}{5.1198}\right)\right]$$

$$= \underline{5.1198}$$

Thus $y(20.2) = \underline{5.1198}$

II Stage: Let $x_0 = 20.2$ $y_0 = 5.1198$

$$f(x, y) = \log_{10}(x/y)$$

$$f(x_0, y_0) = \log_{10}\left(\frac{20.2}{5.1198}\right)$$

$$f(x_0, y_0) = 0.59609$$

$x_1 = x_0 + h = 20.4$, $y(x_1) = y_1 = y(20.4) = ?$

From Euler formula

$$y_1^{(0)} = y_0 + hf(x_0, y_0)$$

$$= 5.1198 + 0.2(0.59609)$$

$$= \underline{5.2390}$$

$$\underline{y_1^{(0)}\ 5.239}$$

By modified Euler's formula

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$= 5.1198 + 0.1\left[0.59609 + \log_{10}\left(\frac{x_1}{y_1^{(0)}}\right)\right]$$

$$= 5.1198 + 0.1\left[0.59609 + \log_{10}\left(\frac{20.4}{5.239}\right)\right]$$

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$y_1^{(1)} = 5.2384$

$y_1^{(2)} = y_0 + h/2 \, [f(x_0, y_0) + f(x_1, y_1^{(1)})]$

$= 5.1198 + 0.1 [0.59609 + \log_{10}(x_1/y_1^{(1)})]$

$= 5.1198 + 0.1 [0.59609 + \log_{10}\left(\frac{20.4}{5.2384}\right)]$

$= 5.2384$

Thus $y(20.4) = 5.2384$

(5) Use modified Euler's method to solve $\frac{dy}{dx} = x + |\sqrt{y}|$ in the range $0 \le x \le 0.4$

June 2017 Dec 18 by taking $h = 0.2$ given that $y = 1$ at $x = 0$ initially.

Solⁿ: we need to find $y(0.2)$ & $y(0.4)$ with $h = 0.2$

I stage: By data $x_0 = 0$ $y_0 = 1$ $f(x, y) = x + \sqrt{y}$, $h = 0.2$ where the modulus sign indicates that we have to take only the +ve value of $\sqrt{y}$.

$f(x_0, y_0) = x_0 + \sqrt{y_0} = 0 + 1 = 1$

$\therefore f(x_0, y_0) = 1$

$x_1 = x_0 + h = 0.2 \Rightarrow x_1 = 0.2$

$y(x_1) = y_1 = y(0.2) = ?$

From

Euler's formula

$y_1^{(0)} = y_0 + hf(x_0, y_0)$

$= 1 + 0.2(1)$

$= 1.2$

By modified Euler's formula

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$= 1 + \frac{0.2}{2}\left[1 + x_1 + \sqrt{y_1^{(0)}}\right]$$

$$= 1 + 0.1\left[1 + 0.2 + \sqrt{1.2}\right]$$

$$= \underline{1.2295}$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$= 1 + 0.1\left[1 + x_1 + \sqrt{y_1^{(1)}}\right]$$

$$= 1 + 0.1\left[1 + 0.2 + \sqrt{1.2295}\right]$$

$$= 1 + 0.1\left[1 + 0.2 + \sqrt{1.2295}\right]$$

$$= \underline{1.23088}$$

$$y_1^{(3)} = y_0 + h/2\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right]$$

$$= 1 + 0.1\left[1 + x_1 + \sqrt{y_1^{(2)}}\right]$$

$$= 1 + 0.1\left[1 + 0.2 + \sqrt{1.23088}\right]$$

$$= \underline{1.2309}$$

$$y_1^{(4)} = y_0 + h/2\left[f(x_0, y_0) + f(x_1, y_1^{(3)})\right]$$

$$= 1 + 0.1\left[1 + 0.2 + \sqrt{1.2309}\right]$$

$$= \underline{1.2309}$$

Thus $y(0.2) = \underline{1.2309}$

II stage: Now Let $x_0 = 0.2 \quad y_0 = 1.2309$

$f(x, y) = x + \sqrt{y}$

$f(x_0, y_0) = x_0 + \sqrt{y_0} = 0.2 + \sqrt{1.2309} = 1.30945$

$f(x_0, y_0) = 1.3095$

$x_1 = x_0 + h$

$x_1 = 0.2 + 0.2 \Rightarrow x_1 = 0.4$

$y(x_1) = y_1 = y(0.4) = ?$

from Eulers formula

$y_1^{(0)} = y_0 + h f(x_0, y_0)$

$= 1.2309 + (0.2)(1.3095)$

$= 1.4928$

from modified Euler's formula

$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$

$= 1.2309 + \frac{0.2}{2}\left[1.3095 + x_1 + \sqrt{y_1^{(0)}}\right]$

$= 1.2309 + 0.1\left[1.3095 + 0.4 + \sqrt{1.4928}\right]$

$= 1.5240$

$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$

$= 1.2309 + 0.1\left[1.3095 + x_1 + \sqrt{y_1^{(1)}}\right]$

$= 1.2309 + 0.1\left[1.3095 + 0.4 + \sqrt{1.5240}\right]$

$= 1.2309 + 0.1\left[1.3095 + 0.4 + \sqrt{1.5240}\right]$

$= 1.5253$

$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right]$

$= 1.2309 + 0.1\left[1.3095 + x_1 + \sqrt{y_1^{(2)}}\right]$

$= 1.2309 + 0.1\left[1.3095 + 0.4 + \sqrt{1.5253}\right]$

$= 1.5253$

thus $y(0.4) = 1.5253$

classmate

Date
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Do yourself

(6) Use modified Euler's method to compute $y(0.1)$ given that $\frac{dy}{dx} = x^2 + y$, $y(0) = 1$ by taking $h = 0.05$ considering the accuracy upto two approximation in each step.

Soln: we need to compute $y(0.05)$ first and use this value to compute $y(0.1)$

I Stage: By data $x_0 = 0$, $y_0 = 1$

$$f(x, y) = x^2 + y, \quad h = 0.05$$

$$f(x_0, y_0) = x_0^2 + y_0 = 0^2 + 1$$

$$f(x_0, y_0) = 1$$

$$x_1 = x_0 + h = 0 + 0.05 \Rightarrow x_1 = 0.05$$

$$y(x_1) = y_1 = y(0.05) = ?$$

From Euler's formula

$$y_1^{(0)} = y_0 + hf(x_0, y_0)$$

$$= 1 + 0.05(1)$$

$$= 1.05$$

From modified Euler's formula

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$= 1 + \frac{0.05}{2}\left[1 + (x_1)^2 + y_1^{(0)}\right]$$

$$= 1 + 0.025\left[1 + (0.05)^2 + 1.05\right]$$

$$y_1^{(1)} = 1.0513$$

$$y_1^{(2)} = 1 + 0.025\left[1 + (0.05)^2 + y_1^{(1)}\right]$$

$$= 1 + 0.025\left[1 + (0.05)^2 + 1.0513\right]$$

$$= 1.0513$$

Thus $y(0.05) = 1.0513$

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II Stage: Now let $x_0 = 0.05$, $y_0 = 1.0513$

$f(x, y) = x^2 + y$;

$f(x_0, y_0) = x_0^2 + y_0 = (0.05)^2 + (1.0513)$

$f(x_0, y_0) = 1.0538$

$x_1 = x_0 + h = 0.05 + 0.05 = 0.1 \Rightarrow x_1 = 0.1$

$y(x_1) = y_1 = y(0.1) = ?$

Now, By Euler's formula

$y_1^{(0)} = y_0 + hf(x_0, y_0)$

$= 1.0513 + 0.05(1.0538)$

$= 1.10399$

Now, from modified Euler's formula

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$= 1.0513 + \frac{0.05}{2}\left[1.0538 + x_1^2 + y_1^{(0)}\right]$$

$$= 1.0513 + 0.025\left[1.0538 + (0.1)^2 + 1.10399\right]$$

$= 1.10389$

$y_1^{(1)} = 1.10389$

$$y_1^{(2)} = 1.0513 + 0.025\left[1.0538 + (0.1)^2 + 1.10389\right]$$

$= 1.10549$

$$y_1^{(3)} = 1.0513 + 0.025\left[1.0538 + (0.1)^2 + 1.10549\right]$$

$= 1.1055$

$$y_1^{(4)} = 1.0513 + 0.025\left[1.0538 + (0.1)^2 + 1.1055\right]$$

$= 1.1055$

Thus $y(0.1) = 1.1055$

Do yourself

(7) Using Euler's predictor and corrector formula solve $\frac{dy}{dx} = x + y$ at $x = 0.2$ given that $y(0) = 1$

Ans: $x_0 = 0,\ y_0 = 1 \quad f(x_0, y_0) = x_0 + y_0 = 1$

$x_1 = x_0 + h = 0.2$

$x_1 = x_0 + h \Rightarrow h = x_1 - x_0$

$h = 0.2 - 0$

$h = 0.2$

$y(x_1) = y_1 = y(0.2) = ?$

we have euler's formula

$$y_1^{(0)} = y_0 + h f(x_0, y_0)$$

$$= 1 + (0.2)(1)$$

$$= 1.2$$

we have from modified Euler's formula

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$= 1 + \frac{0.2}{2}\left[1 + x_1 + y_1^{(0)}\right]$$

$$= 1 + 0.1\left[1 + 0.2 + 1.2\right]$$

$$= 1.24$$

$y_1^{(2)} = 1.244, \quad y_1^{(3)} = 1.2444$

They $y(0.2) = 1.2444$

(8) Using Euler's predictor and corrector formula compute $y(1.1)$, correct to five decimal places given that $\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}$ and $y = 1$ at $x = 1$. Also find the analytical solⁿ.

Solⁿ: By data $x_0 = 1,\ y_0 = 1$

$$\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}$$

$$\frac{dy}{dx} = \frac{1}{x^2} - \frac{y}{x}$$

$$= \frac{1 - yx}{x^2}$$

we have $f(x, y) = \frac{1 - yx}{x^2}$

$$f(x_0, y_0) = \frac{1 - y_0 x_0}{x_0^2} = \frac{1 - (1)(1)}{1^2} = 0$$

$1+h = 1.1 \Rightarrow h = 1.1 - 1 \Rightarrow h = 0.1$

$$x_1 = x_0 + h = 1 + 0.1 = 1.1 \Rightarrow x_1 = 1.1$$

$y(x_1) = y_1 = y(1.1) = ?$

From Euler's formula:

$$y_1^{(0)} = y_0 + h f(x_0, y_0)$$

$$y_1^{(0)} = 1 + (0.1)(0)$$

$$y_1^{(0)} = 1$$

we have modified Euler's formula

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$= 1 + \frac{0.1}{2}\left[0 + \frac{1 - x_1 y_1^{(0)}}{x_1^2}\right]$$

$$= 1 + 0.05\left[\frac{1 - (1.1)(1)}{(1.1)^2}\right]$$

$$y_1^{(1)} = 0.99586$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$= 1 + \frac{0.1}{2}\left[0 + \frac{1 - x_1 y_1^{(1)}}{x_1^2}\right]$$

$$= 1 + 0.05\left[\frac{1 - (1.1)(0.99586)}{(1.1)^2}\right]$$

$$= \underline{0.99605}$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right]$$

$$= 1 + 0.05\left[0 + \frac{1 - x_1 y_1^{(2)}}{x_1^2}\right]$$

$$= 1 + 0.05\left[\frac{1-(1.1)(0.99605)}{(1.1)^2}\right]$$

$$= 0.9960421$$

$$= 0.99605$$

Thus $y(1.1) = 0.99605$

Analytical solution

$\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}$ is of the form $\frac{dy}{dx} + Py = Q$

$P = 1/x$ and $Q = 1/x^2$

whose solution is given by

$$y(IF) = \int Q\, IF\, dx + C$$

where $IF = e^{\int P dx} = e^{\int 1/x\, dx} = e^{\log x} = x$

$\therefore IF = x$

Sol^n is $y(IF) = \int Q\, IF\, dx + C$

$$y(x) = \int \frac{1}{x^2} \times x\, dx + C$$

$$xy = \int \frac{1}{x}\, dx + C$$

$$xy = \log x + C \quad — \circledast$$

Now we have to find $y(1.1)$

by using the initial condition

$x = 1 \quad y = 1$

$(1)(1) = \log(1) + C \quad \Rightarrow 1 = 0 + C \Rightarrow C = 1$

$[\log(1) = 0]$

put c value in (*)

$xy = \log x + 1$

or $y = \frac{1+\log x}{x}$

put $x = 1.1$ we will get $y(1.1)$

$y = \frac{1+\log(1.1)}{1.1}$

$y = \underline{0.995736}$ [use ln]

$y = \underline{0.99574}$ is the analytical solution

Runge Kutta method of fourth order

Consider the initial value problem

$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$

we need to find $y(x_0+h)$ where h is the step size.

we have to first compute K_1, K_2, K_3, K_4 by the following formula

$K_1 = hf(x_0, y_0)$

$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$

$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right)$

$K_4 = hf(x_0 + h, y_0 + K_3)$

The required

$y(x_0+h) = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

Problems

1) Given $\frac{dy}{dx} = 3x + \frac{y}{2}$, $y(0) = 1$, compute $y(0.2)$ by taking $h = 0.2$ using Range-Kutta method of fourth order. Also find the analytical solution.

Solⁿ: By data $f(x, y) = \frac{dy}{dx} = 3x + \frac{y}{2}$

$$\therefore f(x, y) = 3x + \frac{y}{2},$$

$x_0 = 0 \quad y_0 = 1 \quad h = 0.2 \quad f(x_0, y_0) = 0.5$

we shall first find K_1, K_2, K_3, K_4

$K_1 = hf(x_0, y_0) = (0.2) f(0, 1)$

$K_1 = (0.2)(0.5)$

$K_1 = 0.1$

$K_2 = hf(x_0 + h/2, y_0 + K_1/2)$

$= 0.2 f\left(0 + \frac{0.2}{2}, 1 + \frac{0.1}{2}\right)$

$= 0.2 f(0.1, 1.05)$

$= 0.2 \left[3(0.1) + \frac{1.05}{2}\right]$

$= 0.165$

$K_3 = hf(x_0 + h/2, y_0 + K_2/2)$

$= 0.2 f\left(0 + \frac{0.2}{2}, 1 + \frac{0.165}{2}\right)$

$= 0.2 f(0.1, 1.0825)$

$= 0.2 \left[3(0.1) + \frac{1.0825}{2}\right]$

$K_3 = 0.16825$

$K_4 = hf(x_0+h,\ y_0+K_3)$
$= 0.2f(0+0.2,\ 1+0.16825)$

$= 0.2f(0.2,\ 1.16825)$

$= 0.2\left[3(0.2) + \frac{1.16825}{2}\right]$

$K_4 = 0.236825$

$y(x_0+h) = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

$y(0.2) = 1 + \frac{1}{6}(0.1 + 2(0.165) + 2(0.16825) + 0.236825)$

$= 1.1672$

we shall find the analytical solution of the given equation by writing in the form $\frac{dy}{dx} + Py = Q$ whose solution is

$y(IF) = \int Q\,IF\,dx + c$

where $IF = e^{\int P.dx}$ where $\frac{dy}{dx} - \frac{y}{2} = 3x$

here $P = -\frac{1}{2}$ $Q = 3x$

$IF = e^{\int P.dx} = e^{\int -\frac{1}{2}dx}$

$IF = e^{-\frac{1}{2}x}$

Soln is $ye^{-\frac{1}{2}x} = \int 3x e^{-x/2} dx + c$

$ye^{-x/2} = 3\int x e^{-x/2} dx + c$

Integrating RHS by parts we have

$$ye^{-x/2} = 3\left[x\frac{e^{-x/2}}{-1/2} - \int \frac{e^{-x/2}}{-1/2}(1)\,dx\right] + c$$

$$= 3\left[-2xe^{-x/2} + 2\frac{e^{-x/2}}{-1/2}\right] + c$$

$$ye^{-x/2} = -6xe^{-x/2} - 12e^{-x/2} + c$$

$\div$ B.S by $e^{-x/2}$

$$y = -6x\cancel{\frac{e^{-x/2}}{e^{-x/2}}} - 12\cancel{\frac{e^{-x/2}}{e^{-x/2}}} + \frac{c}{e^{-x/2}}$$

$$y = -6x\cancel{e^{-x/2}}\cdot\cancel{e^{x/2}} - 12 + ce^{x/2}$$

$$\therefore y = -6x - 12 + ce^{x/2}$$

Use initial condition to find c, $x=0, y=1$, $1 = 0-12+ce^{0}$

$\Rightarrow$ $c = 13$, now by putting $x = 0.2$ we have

$$y(0.2) = -6(0.2) - 12 + 13e^{0.2/2}$$

$$y(0.2) = 1.1672219$$

$y(0.2) = 1.1672$ by analytical solution

(2) Do yourself

Use fourth order Runge Kutta method to solve $(x+y)\frac{dy}{dx} = 1$, $y(0.4) = 1$ at $x = 0.5$ correct to four decimal places

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Solⁿ: $\frac{dy}{dx} = \frac{1}{x+y}$, $y_0 = 1$, $x_0 = 0.4$, $y(0.5) = ?$

$x = x_0 + h$

$x = 0.4 + h$

$0.5 - 0.4 = h \Rightarrow h = 0.1$

$K_1 = 0.0714$

$K_2 = 0.0673$

$K_3 = 0.0674$

$K_4 = 0.0638$

$y(0.5) = 1.0674$ //

DEC17 June 18 (3) Using Runge Kutta method of fourth order find $y(0.2)$ for the equation $\frac{dy}{dx} = \frac{y-x}{y+x}$

$y(0) = 1$ taking $h = 0.2$

Solⁿ: By data

$f(x, y) = \frac{y-x}{y+x}$ $x_0 = 0$ $y_0 = 1$ $h = 0.2$

$y(0.2) = ?$

we shall find K_1, K_2, K_3, K_4

$K_1 = hf(x_0, y_0) = (0.2) f(0, 1)$

$= (0.2)\left[\frac{1-0}{1+0}\right]$

$K_1 = 0.2$

$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$

$= (0.2) f\left(0 + \frac{0.2}{2}, 1 + \frac{0.2}{2}\right)$

$= (0.2) f(0.1, 1 + 0.1)$

$= (0.2) f(0.1, 1.1)$

$$K_2 = (0.2)\left[\frac{1.1-0.1}{1.1+0.1}\right]$$

$$K_2 = 0.1667$$

$$K_3 = hf\left(x_0+h/2,\ y_0+\frac{K_2}{2}\right)$$

$$= 0.2f\left(0+\frac{0.2}{2},\ 1+\frac{0.1667}{2}\right)$$

$$= 0.2f(0.1,\ 1.08335)$$

$$= 0.2\left[\frac{1.08335-0.1}{1.08335+0.1}\right]$$

$$= 0.2\left[\frac{0.98335}{1.18335}\right]$$

$$= 0.16619$$

$$K_4 = hf(x_0+h,\ y_0+K_3)$$

$$= 0.2f(0+0.2,\ 1+0.16619)$$

$$= 0.2f(0.2,\ 1.16619)$$

$$= 0.2\left[\frac{1.16619-0.2}{1.16619+0.2}\right]$$

$$= 0.1414$$

$$y(x_0+h) = y_0 + 1/6\,(K_1+2K_2+2K_3+K_4)$$

$$y(0.2) = 1 + 1/6\,(0.2+2(0.1667)+2(0.16619)+0.1414)$$

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$y(0.2) = 1.16786$

(ii) Use fourth order Runge Kutta method to find y at $x = 0.1$ given that $\frac{dy}{dx} = 3e^{x} + 2y$, $y(0) = 0$ and $h = 0.1$

Sol^n: By data $f(x, y) = 3e^{x} + 2y$, $x_0 = 0$, $y_0 = 0$, $h = 0.1$

$y(0.1) = ?$

$K_1 = h f(x_0, y_0) = 0.1 f(0, 0) = (0.1)(3e^{0} + 2(0))$

$K_1 = (0.1)(3)$

$K_1 = 0.3$ //

$K_2 = h f(x_0 + h/2, y_0 + K_1/2)$

$= 0.1 f\left(0 + \frac{0.1}{2}, 0 + \frac{0.3}{2}\right)$

$= 0.1 f(0.05, 0.15)$

$= 0.1 [3e^{0.05} + 2(0.15)]$

$= 0.34538$

$K_3 = h f(x_0 + h/2, y_0 + K_2/2)$

$= 0.1 f\left(0.05, 0 + \frac{0.34538}{2}\right)$

$= 0.1 f(0.05, 0.17269)$

$= 0.1 [3e^{0.05} + 2(0.17269)]$

$= 0.3499$ //

$K_4 = h f(x_0 + h, y_0 + K_3)$

$= 0.1 f(0.1, 0 + 0.3499)$

~~= 0.1 f (0.05, 0.17499)~~

~~= 0.1 [3e^0.95 + 2(0.17495)]~~

$= 0.1\left[3e^{0.01} + 2(0.3499)\right]$

$= 0.4015$

$y(x_0+h) = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

$= 0 + \frac{1}{6}(0.3 + 2(0.3454) + 2(0.3499) + 0.4015)$

$y(0.1) = 0.34868$

Do yourself

(5) Use fourth order Runge Kutta method to find $y(1.1)$ given that $\frac{dy}{dx} = xy^{1/3}$, $y(1) = 1$

Sol^n: By data $f(x, y) = xy^{1/3}$ $x_0 = 1$, $y_0 = 1$

we need to find $y(1.1)$

$x_0 + h = 1.1$

$h = 1.1 - x_0$

$h = 1.1 - 1$

$h = 0.1$

we shall find K_1, K_2, K_3, K_4

$K_1 = 0.1$

$K_2 = 0.1067$ $\qquad y(1.1) = 1.1068$

$K_3 = 0.1068$

$K_4 = 0.1138$

Do yourself

(6) Using Runge Kutta method of fourth order solve $\frac{dy}{dx} + y = 2x$ at $x = 1.1$ given that $y = 3$ at $x = 1$ initially

By data

Sol^n: $\frac{dy}{dx} = 2x - y$, $y_0 = 3$ $x_0 = 1$

$f(x,y) = 2x - y$, $x_0 + h = 1.1 \Rightarrow h = 1.1 - x_0 = 1.1 - 1$
$h = 0.1$//

we shall find K_1, K_2, K_3, K_4

$K_1 = hf(x_0, y_0) = (0.1) f(1,3) = 0.1[2(1) - 3]$
$K_1 = -0.1$

$K_2 = hf(x_0 + h/2, y_0 + K_1/2)$

$= (0.1) f\left(1 + \frac{0.1}{2}, 3 + \frac{(-0.1)}{2}\right)$

$= (0.1) f(1.05, 2.95)$
$= (0.1)(2(1.05) - 2.95)$
$= -0.085$//

$K_3 = hf(x_0 + h/2, y_0 + K_2/2)$

$= 0.1 f\left(1.05, 3 + \frac{(-0.085)}{2}\right)$

$= 0.1 f(1.05, 2.9575)$

$= 0.1 [2(1.05) - 2.9575]$
$= -0.08575$//

$K_4 = hf(x_0 + h, y_0 + K_3)$
$= 0.1 f(1 + 0.1, 3 + (-0.08575))$
$= 0.1 f(1.1, 3 + (-0.08575))$

$= 0.1 f(1.1, 2.91425)$
$= 0.1 [2(1.1) + 2.91425]$

$K_4 = \underline{\underline{-0.071425}}$

$$y(x_0+h) = y_0 + \tfrac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

$$= 3 + \frac{1}{6}\left(-0.1 + 2(-0.085) + 2(-0.08575) - 0.071425\right)$$

$$y(1.1) = 2.9145125 \simeq \underline{2.9145}$$

(7) Using Runge Kutta method of fourth order find $y(0.2)$ for the equation

Dec 2018 $\frac{dy}{dx} = \frac{y-x}{y+x}$, $y(0)=1$ taking $h=0.1$

Sol^n: The problem has to be done in two stages

I Stage: $f(x,y) = \frac{y-x}{y+x}$, $x_0=0$, $y_0=1$, $h=0.1$

we have to find $y(0.1)$

we shall find K_1, K_2, K_3, K_4

$$K_1 = hf(x_0, y_0)$$

$$= (0.1) f(0,1)$$

$$= (0.1)\left[\frac{1-0}{1+0}\right]$$

$$= (0.1)$$

$$K_1 = 0.1$$

$$K_2 = hf\left(x_0 + h/2,\ y_0 + K_1/2\right)$$

$$= (0.1) f(0+0.05,\ 1+0.05)$$

$$= (0.1) f(0.05,\ 1.05)$$

$$= (0.1)\left[\frac{1.05-0.05}{1.05+0.05}\right]$$

$$= \underline{0.0909}$$

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$K_3 = hf(x_0 + h/2,\ y_0 + K_2/2)$

$= (0.1)\, f\left(0 + 0.05,\ 1 + \frac{0.0909}{2}\right)$

$= (0.1)\, f(0.05,\ 1.04545)$

$= (0.1)\left[\frac{1.04545 - 0.05}{1.04545 + 0.05}\right]$

$= 0.09087$

$K_4 = hf(x_0 + h,\ y_0 + K_3)$

$= (0.1)\, f(0 + 0.1,\ 1 + 0.09087)$

$= (0.1)\, f(0.1,\ 1.09087)$

$= 0.0832$

$y(x_0 + h) = y_0 + 1/6\,(K_1 + 2K_2 + 2K_3 + K_4)$

$= 1 + 1/6\,(0.1 + 2(0.0909) + 2(0.09087) + 0.0832)$

$y(0.1) = 1.09112$

II Stage: $f(x, y) = \frac{y - x}{y + x}$, $x_0 = 0.1$ $y_0 = 1.09112$ $h = 0.1$

we need to find $y(0.2)$

$K_1 = hf(x_0, y_0)$

$= (0.1)\, f(0.1,\ 1.09112)$

$= 0.1\left[\frac{1.09112 - 0.1}{1.09112 + 0.1}\right]$

$= 0.0832$

$K_2 = hf(x_0 + h/2,\ y_0 + K_1/2)$

$= (0.1)\, f\left(0.1 + \frac{0.1}{2},\ 1.09112 + \frac{0.0832}{2}\right)$

$= (0.1)\, f(0.15,\ 1.13272)$

$= (0.1)\left[\frac{1.13272 - 0.15}{1.13272 + 0.15}\right]$

$= 0.0766$

$K_3 = hf(x_0 + h/2,\ y_0 + K_2/2)$

$= (0.1)\, f\left(0.1 + \frac{0.1}{2},\ 1.09112 + \frac{0.0766}{2}\right)$

$= (0.1)\, f(0.15,\ 1.12942)$

$= (0.1) f(0.15, 1.12942)$

$= (0.1) \left[\dfrac{1.12942 - 0.15}{1.12942 + 0.15} \right]$

$= 0.07655$

$K_4 = hf(x_0 + h, y_0 + K_3)$

$= (0.1) f(0.1 + 0.1, 1.09112 + 0.07655)$

$= (0.1) f(0.2, 1.16767)$

$= (0.1) \left[\dfrac{1.16767 - 0.2}{1.16767 + 0.2} \right]$

$K_4 = 0.07075$

$y(x_0 + h) = y_0 + \frac{1}{6} (K_1 + 2K_2 + 2K_3 + K_4)$

$= 1.09112 + \frac{1}{6} (0.0832 + 2(0.0766) + 2(0.07655) + 0.07075)$

$y(0.2) = 1.167828$

8. Solve: $(y^2 - x^2)\, dx = (y^2 + x^2)\, dy$ for $x = 0.2$ and 0.4 given that $y = 1$ at $x = 0$ initially, by applying Runge Kutta method of order 4.

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Soln: we have $\dfrac{dy}{dx} = \dfrac{y^2 - x^2}{y^2 + x^2}$ $x_0 = 0$, $y_0 = 1$ $h = 0.2$

here we have to find $y(0.2)$

I Stage: $f(x, y) = \dfrac{y^2 - x^2}{y^2 + x^2}$ we shall find K_1, K_2, K_3, K_4

$K_1 = hf(x_0, y_0) = (0.2) f(0, 1)$

$= (0.2) \left[\dfrac{1^2 - 0^2}{1^2 + 0^2} \right]$

$K_1 = (0.2)$

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$K_2 = hf(x_0+h/2,\ y_0+K_1/2)$

$= (0.2) f\left(0+\frac{0.2}{2},\ 1+\frac{0.2}{2}\right)$

$= (0.2) f(0.1,\ 1.1)$

$= (0.2)\left[\frac{(1.1)^2-(0.1)^2}{(1.1)^2+(0.1)^2}\right]$

$= \underline{0.19672}$

$K_3 = hf(x_0+h/2,\ y_0+K_2/2)$

$= (0.2) f\left(0+\frac{0.2}{2},\ 1+\frac{0.19672}{2}\right)$

$= (0.2) f(0.1,\ 1.09836)$

$= (0.2)\left[\frac{(1.09836)^2-(0.1)^2}{(1.09836)^2+(0.1)^2}\right]$

$= \underline{0.19671}$

$K_4 = hf(x_0+h,\ y_0+K_3)$

$= (0.2) f(0+0.2,\ 1+0.19671)$

$= (0.2) f(0.2,\ 1.19671)$

$= (0.2)\left[\frac{(1.19671)^2-(0.2)^2}{(1.19671)^2+(0.2)^2}\right]$

$= 0.18913$

we have $y(x_0+h) = y_0 + 1/6\,(K_1+2K_2+2K_3+K_4)$

$= 1 + \frac{1}{6}\,(0.2 + 2(0.19672) + 2(0.19671) + 0.18913)$

$y(0.2) = \underline{1.1959}$

II Stage: Now we have to find $y(0.4)$

$f(x,y) = \frac{y^2-x^2}{y^2+x^2}$ $\quad x_0 = 0.2,\ y_0 = 1.1959,\ h = 0.2$

$K_1 = hf(x_0+y_0)$

$= (0.2) f(0.2,\ 1.1959)$

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$= 0.2 f(0.2, 1.1959)$

$$= (0.2)\left[\frac{(1.1959)^2 - (0.2)^2}{(1.1959)^2 + (0.2)^2}\right]$$

$= 0.1891$

$k_2 = hf(x_0 + h/2, y_0 + k_1/2)$

$$= (0.2) f\left(0.2 + \frac{0.2}{2}, 1.1959 + \frac{0.1891}{2}\right)$$

$= (0.2) f(0.3, 1.29045)$

$$= (0.2)\left[\frac{(1.29045)^2 - (0.3)^2}{(1.29045)^2 + (0.3)^2}\right]$$

$= 0.17949$

$k_3 = hf(x_0 + h/2, y_0 + k_2/2)$

$$= (0.2) f\left(0.2 + \frac{0.2}{2}, 1.1959 + \frac{0.17949}{2}\right)$$

$= (0.2) f(0.3, 1.285645)$

$$= (0.2)\left[\frac{(1.285645)^2 - (0.3)^2}{(1.285645)^2 + (0.3)^2}\right]$$

$= 0.17934$

$k_4 = hf(x_0 + h, y_0 + k_3)$

$= (0.2) f(0.2 + 0.2, 1.1959 + 0.17934)$

$= (0.2) f(0.4, 1.37524)$

$$= (0.2)\left[\frac{(1.37524)^2 - (0.4)^2}{(1.37524)^2 + (0.4)^2}\right]$$

$= 0.1688$

$y(x_0+h) = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

$= 1.1959 + \frac{1}{6}(0.1891 + 2(0.17949) + 2(0.17934) + 0.1688)$

$y(0.4) = \underline{1.37516}$

Numerical Predictor and Corrector Methods

we discuss two predictor and corrector methods namely

① Milne's method ② Adams - Bashforth method

Consider the differential equation $y' = \frac{dy}{dx} = f(x,y)$ with a set of four predetermined values of y; $y(x_0) = y_0$, $y(x_1) = y_1$, $y(x_2) = y_2$ and $y(x_3) = y_3$

here x_0, x_1, x_2, x_3 are equally spaced value of x with width h

Predictor and corrector formula to compute $y(x_4) = y_4$ are as follows

Milne's predictor and Corrector formula

$$y_4^{(P)} = y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3') \text{... (predictor formula)}$$

$$y_4^{(c)} = y_2 + \frac{h}{3}(y_2' + 4y_3' + y_4') \text{... (corrector formula)}$$

Adams - Bashforth predictor and corrector formula

$$y_4^{(P)} = y_3 + \frac{h}{24}(55y_3' - 59y_2' + 37y_1' - 9y_0') \text{ (predictor formula)}$$

$$y_4^{(c)} = y_3 + \frac{h}{24}(9y_4' + 19y_3' - 5y_2' + y_1') \text{ (corrector formula)}$$

working procedure

1. we first prepare the table showing the values of y corresponding to four equidistant values of x and the computation of $y' = f(x, y)$
2. we compute y_4 from the predictor formula
3. we use this value of y_4 to compute $y'_4 = f(x_4, y_4)$
4. we apply corrector formula to obtain the corrected value of y_4
5. This value is used for computing y'_4 to apply the corrector formula again
6. The process is continued till we get consistency in two consecutive values of y_4

NOTE: we can also find $y_5, y_6 \ldots$ by deducing expressions from the general form of predictor and corrector formula.

Problems

1. Given that $\frac{dy}{dx} = x - y^2$ and the data $y(0) = 0$,

June 2018 $y(0.2) = 0.02$, $y(0.4) = 0.0795$, $y(0.6) = 0.1762$
Compute y at $x = 0.8$ by applying
   1. milne's method
   2. Adams - Bashforth method

Solⁿ: we prepare the following table using the given data which is essentially required for applying the predictor and corrector formula

P.T.O

x	y	$y' = x - y^2$
$x_0 = 0$	$y_0 = 0$	$y'_0 = 0 - 0^2 = 0$
$x_1 = 0.2$	$y_1 = 0.02$	$y'_1 = 0.2 - (0.02)^2 = 0.1996$
$x_2 = 0.4$	$y_2 = 0.0795$	$y'_2 = 0.4 - (0.0795)^2 = 0.3937$
$x_3 = 0.6$	$y_3 = 0.1762$	$y'_3 = 0.6 - (0.1762)^2 = 0.5689$
$x_4 = 0.8$	$y_4 = ?$	

(a) By milne's method

we have the predictor formula

$$y_4^{(P)} = y_0 + \frac{4h}{3}(2y'_1 - y'_2 + 2y'_3)$$

$$= 0 + \frac{4(0.2)}{3}(2(0.1996) - 0.3937 + 2(0.5689))$$

$$= 0.30488$$

$$y'_4 = x_4 - y_4^2$$

$$= 0.8 - (0.30488)^2$$

$$= 0.707$$

we have the corrector formula

$$y_4^{(C)} = y_2 + h/3\,(y'_2 + 4y'_3 + y'_4)$$

$$= 0.0795 + \frac{0.2}{3}(0.3937 + 4(0.5689) + 0.707)$$

$$= 0.30458$$

Now, $y'_4 = x_4 - y_4^2$

$$= 0.8 - (0.30458)^2$$

$$= 0.7072$$

Subytituting the value of y'_4 again in the Corrector formula

$$y_4^{(C)} = 0.0795 + \frac{0.2}{3}[0.3937 + 4(0.5689) + 0.7072]$$

$$y_4^{(C)} = 0.3046$$

$\therefore \ y_4 = y(0.8) = 0.3046$

(b) By Adams - Bashforth method

we have predictor formula

$$y_4^{(P)} = y_3 + \frac{h}{24}(55y_3' - 59y_2' + 37y_1' - 9y_0')$$

$$= 0.1762 + \frac{0.2}{24}\left[55(0.5689) - 59(0.3937) + 37(0.1996) - 9(0)\right]$$

$$y_4^{(P)} = 0.30492$$

Now, $y_4' = x_4 - y_4^2$

$$y_4' = 0.8 - (0.30492)^2$$

$$y_4' = 0.707$$

Next, we have the corrector formula

$$y_4^{(c)} = y_3 + \frac{h}{24}(9y_4' + 19y_3' - 5y_2' + y_1')$$

$$y_4^{(c)} = 0.1762 + \frac{0.2}{24}\left[9(0.707) + 19(0.5689) - 5(0.3937) + 0.1996\right]$$

$$y_4^{(c)} = 0.30456$$

$$y_4' = x_4 - y_4^2$$

$$= 0.8 - (0.30456)^2$$

$$\simeq 0.7072$$

Applying corrector formula again with only change in the value of y_4' we obtain,

$$y_4^{(c)} = 0.1762 + \frac{0.2}{24}\left[9(0.7072) + 19(0.5689) - 5(0.3937) + 0.1996\right]$$

$y_4^{(1)} = 0.30456$

Thus $y_4 = y(0.8) = 0.30456$

(8) Apply milne's method to compute $y(1.4)$ correct to four decimal places given $\frac{dy}{dx} = x^2 + y/2$ and following data $y(1) = 2$, $y(1.1) = 2.2156$, $y(1.2) = 2.4649$, $y(1.3) = 2.7514$

Soln: First we shall prepare the following table

x	y	$y' = x^2 + y/2$
$x_0 = 1$	$y_0 = 2$	$y_0' = 1^2 + 2/2 = 1+1 = 2$
$x_1 = 1.1$	$y_1 = 2.2156$	$y_1' = (1.1)^2 + \frac{2.2156}{2} = 2.3178$
$x_2 = 1.2$	$y_2 = 2.4649$	$y_2' = (1.2)^2 + \frac{2.4649}{2} = 2.67245$
$x_3 = 1.3$	$y_3 = 2.7514$	$y_3' = (1.3)^2 + \frac{2.7514}{2} = 3.0657$
$x_4 = 1.4$	$y_4 = ?$	

we have $y_4^{(P)} = y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3')$

$$y_4^{(P)} = 2 + \frac{4(0.1)}{3}\left[2(2.3178) - 2.67245 + 2(3.0657)\right]$$

$$= 3.07927$$

$$\therefore y_4' = x_4^2 + \frac{y_4}{2}$$

$$= (1.4)^2 + \frac{3.07927}{2}$$

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$y_4' = 3.49963$

Now consider

$$y_4^{(c)} = y_2 + \frac{h}{3}(y_2' + 4y_3' + y_4')$$

$$= 2.4649 + \frac{0.1}{3}\left[2.67245 + 4(3.0657) + 3.49963\right]$$

$$y_4^{(c)} = 3.07939$$

Now, $y_4' = x_4^2 + \frac{y_4}{2}$

$$= (1.4)^2 + \frac{3.07939}{2}$$

$$= 3.49969$$

Substituting this value of y_4' again in the corrector formula we obtain

$$y_4^{(c)} = 2.4649 + \frac{0.1}{3}\left[2.67245 + 4(3.0657) + 3.49969\right]$$

$$y_4^{(c)} = 3.07939$$

Thus $y_4 = y(1.4) = 3.07939$

③ If $\frac{dy}{dx} = 2e^x - y$, $y(0) = 2$, $y(0.1) = 2.010$, $y(0.2) = 2.040$ and $y(0.3) = 2.090$ find $y(0.4)$ correct to four decimal places by using

ⓐ milne's predictor - corrector method

ⓑ Adams - Bashforth predictor - corrector method

(Apply the corrector formula twice)

x	y	$y' = 2e^{x} - y$
$x_0 = 0$	$y_0 = 2$	$y_0' = 2e^{0} - 2 = 0$
$x_1 = 0.1$	$y_1 = 2.010$	$y_1' = 2e^{x_1} - y_1 = 2e^{0.1} - 2.010 = 0.2003$
$x_2 = 0.2$	$y_2 = 2.040$	$y_2' = 2e^{x_2} - y_2 = 2e^{0.2} - 2.040 = 0.4028$
$x_3 = 0.3$	$y_3 = 2.090$	$y_3' = 2e^{x_3} - y_3 = 2e^{0.3} - 2.090 = 0.6097$
$x_4 = 0.4$	$y_4 = ?$	

(a) By milne's predictor - corrector method

$$y_4^{(P)} = y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3')$$

$$= 2 + \frac{4(0.1)}{3}\left(2(0.2003) - (0.4028) + 2(0.6097)\right)$$

$$= 2.16229$$

$$y_4^{(P)} = 2.1623$$

Now, $y_4' = 2e^{x_4} - y_4$

$$= 2e^{0.4} - 2.1623$$

$$= 0.8213$$

Next, we have milne's corrector formula

$$y_4^{(c)} = y_2 + h/3\,(y_2' + 4y_3' + y_4')$$

$$= 2.040 + \frac{0.1}{3}(0.4028 + 4(0.6097) + 0.8213)$$

$$= 2.16209$$

$$y_4^{(c)} = 2.1621$$

$$y_4' = 2e^{x_4} - y_4$$

$$= 2e^{0.4} - 2.1621$$

$y'_4 = 0.8215$

Applying corrector formula again we have

$$y_4^{(c)} = 2.04 + \frac{0.1}{3}\left[0.4028 + 4(0.6097) + 0.8215\right]$$

$$= 2.1621$$

Thus $y(0.4) = 2.1621$

(b) By Adams-Bashforth predictor-Corrector method

we have $y_4^{(P)} = y_3 + h/_{24}\,(55y'_3 - 59y'_2 + 37y'_1 - 9y'_0)$

$$y_4^{(P)} = 2.090 + \frac{0.1}{24}\left[55(0.6097) - 59(0.4028) + 37(0.2003) - 9(0)\right]$$

$$y_4^{(P)} = 2.16158$$

$$y_4^{(P)} = 2.1616$$

Now, $y'_4 = 2e^{x_4} - y_4$

$$y'_4 = 2e^{0.4} - 2.1616$$

$$y'_4 = 0.822$$

Next we have, $y_4^{(c)} = y_3 + h/_{24}\,(9y'_4 + 19y'_3 - 5y'_2 + y'_1)$

$$y_4^{(c)} = 2.090 + \frac{0.1}{24}\left[9(0.822) + 19(0.6097) - 5(0.4028) + 0.2003\right]$$

$$y_4^{(c)} = 2.1615$$

Now, $y'_4 = 2e^{0.4} - y_4$

$$y'_4 = 2e^{0.4} - 2.1615$$

$$= 0.822149 \Rightarrow y'_4 = 0.82215$$

Substituting again in the corrector formula, we obtain $y_4^{(1)} = 2.090 + \frac{0.1}{24}\left[9(0.88215) + 19(0.6097) - 5(0.4028) + 0.2003\right]$

$$y_4^{(1)} = 2.1615$$

Thus $y(0.4) = 2.1615$

(4) Apply Adams-Bashforth method to solve the equation $(y^2+1)\,dy - x^2\,dx = 0$ at $x = 1$ given $y(0) = 1$, $y(0.25) = 1.0026$, $y(0.5) = 1.0206$, $y(0.75) = 1.0679$. Apply corrector formula twice.

Soln: By data $\frac{dy}{dx} = y' = \frac{x^2}{y^2+1}$

we prepare the following table

x	y	$y' = \frac{x^2}{y^2+1}$
$x_0 = 0$	$y_0 = 1$	$y_0' = \frac{x_0^2}{y_0^2+1} = \frac{0}{1+1} = 0$
$x_1 = 0.25$	$y_1 = 1.0026$	$y_1' = \frac{x_1^2}{y_1^2+1} = \frac{(0.25)^2}{(1.0026)^2+1} = 0.03116$
$x_2 = 0.5$	$y_2 = 1.0206$	$y_2' = \frac{x_2^2}{y_2^2+1} = \frac{(0.5)^2}{(1.0206)^2+1} = 0.12245$
$x_3 = 0.75$	$y_3 = 1.0679$	$y_3' = \frac{x_3^2}{y_3^2+1} = \frac{(0.75)^2}{(1.0679)^2+1} = 0.2628$
$x_4 = 1$	$y_4 = ?$	

we have the predictor formula

$$y_4^{(P)} = y_3 + \frac{h}{24}\left[55y_3' - 59y_2' + 37y_1' - 9y_0'\right]$$

$$y_4^{(P)} = 1.0679 + \frac{0.25}{24}\left[55(0.2628) - 59(0.12245) + 37(0.03116) - 9(0)\right]$$

$$= 1.1552$$

$$y_4' = \frac{x_4^2}{y_4^2 + 1}$$

$$y_4' = \frac{1^2}{(1.1552)^2 + 1}$$

$$y_4' = 0.42835$$

$$y_4' = 0.4284$$

Next, we have the corrector formula

$$y_4^{(c)} = y_3 + h/24\,(9y_4' + 19y_3' - 5y_2' + y_1')$$

$$= 1.0679 + \frac{0.25}{24}\left[9(0.4284) + 19(0.2628) - 5(0.12245) + 0\right]$$

$$= 1.1536$$

$$y_4^{(c)} = 1.154$$

Now, $$y_4' = \frac{1^2}{(1.154)^2 + 1}$$

$$y_4' = 0.4288$$

Applying the corrector formula again we obtain

$$y_4^{(c)} = 1.0679 + \frac{0.25}{24}\left[9(0.4288) + 19(0.2628) - 5(0.12245) + 0\right]$$

$$y_4^{(c)} = 1.1537 \Rightarrow y_4^{(c)} = 1.154$$

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Thus $y(1) = 1.154$

⑥ The following table gives the solution of $5xy' + y^2 - 2 = 0$. Find the value of y at $x = 4.5$ using Milne's predictor and corrector formula. Use the corrector formula twice.

x	4	4.1	4.2	4.3	4.4	//////
y	1	1.0049	1.0097	1.0143	1.0187	//////

Soln: By data

$$5xy' + y^2 - 2 = 0$$

$$5xy' = 2 - y^2$$

$$y' = \frac{2 - y^2}{5x}$$

x	y	$y' = \frac{2-y^2}{5x}$
$x_0 = 4$	$y_0 = 1$	$y_0' = \frac{2 - y_0^2}{5x_0} = \frac{2-(1)^2}{5(4)} = 0.05$
$x_1 = 4.1$	$y_1 = 1.0049$	$y_1' = \frac{2 - y_1^2}{5x_1} = \frac{2-(1.0049)^2}{5(4.1)} = 0.047829$
$x_2 = 4.2$	$y_2 = 1.0097$	$y_2' = \frac{2 - y_2^2}{5x_2} = \frac{2-(1.0097)^2}{5(4.2)} = 0.04669$
$x_3 = 4.3$	$y_3 = 1.0143$	$y_3' = \frac{2 - y_3^2}{5x_3} = \frac{2-(1.0143)^2}{5(4.3)} = 0.04415$ 0.04517
$x_4 = 4.4$	$y_4 = 1.0187$	$y_4' = \frac{2 - y_4^2}{5x_4} = \frac{2-(1.0187)^2}{5(4.4)} = 0.0437$
$x_5 = 4.5$	$y_5 = ?$	

We have Milne's predictor and corrector formula in the standard form

$$y_4^{(P)} = y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3');$$

$$y_4^{(C)} = y_2 + h/3\,(y_2' + 4y_3' + y_4')$$

Since we require y_5, the equivalent form of these formula are given by

$$y_5^{(P)} = y_1 + \frac{4h}{3}\left[2y_2' - y_3' + 2y_4'\right]$$

$$y_5^{(C)} = y_3 + h/3\left[y_3' + 4y_4' + y_5'\right]$$

hence $y_5^{(P)} = 1.0049 + \frac{4\,(0.1)}{3}\left[2(0.04669) - 0.04415 + 2\times 0.04373\right]$

$$= \underline{\underline{1.02312}}$$

$$y_5' = \frac{2 - y_5^2}{5x_5}$$

$$= \frac{2 - (1.02312)^2}{5(4.5)}$$

$$= \underline{\underline{0.04236}}$$

hence, $y_5^{(C)} = y_3 + h/3\left[y_3' + 4y_4' + y_5'\right]$

$$= 1.0143 + \frac{0.1}{3}\left[\overset{0.04517}{\cancel{0.04415}} + 4(0.04373) + 0.04236\right]$$

$$= 1.023048$$

$$\simeq \underline{\underline{1.023}}$$

$$y_5' = \frac{2 - y_5^2}{5x_5}$$

$$= \frac{2 - (1.023)^2}{5(4.5)}$$

$$y_5' = \underline{\underline{0.04237}}$$

applying corrector formula again

$$y_5^{(1)} = y_3 + \frac{h}{3}\left[y_3' + 4y_4' + y_5'\right]$$

$$= 1.0143 + \frac{0.1}{3}\left[\cancel{0.04415}\ 0.04517 + 4(0.04373) + 0.04237\right]$$

$$= 1.023048$$

$$\approx 1.023$$

Thus $\underline{y(4.5) = 1.023}$

⑥ Solve the differential equation $y' + y + xy^2 = 0$ with the initial value of $y: y_0 = 1,\ y_1 = 0.9008,\ y_2 = 0.8066,\ y_3 = 0.722$ corresponding to the values of $x: x_0 = 0,\ x_1 = 0.1,\ x_2 = 0.2,\ x_3 = 0.3$ by computing the value of y corresponding to $x = 0.4$. Applying Adams - Bashforth predictor and corrector formula

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Sol^{n}:

x	y	$y' = -(y + xy^2)$
$x_0 = 0$	$y_0 = 1$	$y_0' = -(y_0 + x_0 y_0^2) = -(1 + (0)(1)) = -1$
$x_1 = 0.1$	$y_1 = 0.9008$	$y_1' = -(y_1 + x_1 y_1^2) = -(0.9008 + (0.1)(0.9008)^2) = -0.9819$
$x_2 = 0.2$	$y_2 = 0.8066$	$y_2' = -(0.8066 + (0.2)(0.8066)^2) = -0.9367$
$x_3 = 0.3$	$y_3 = 0.722$	$y_3' = -(0.722 + (0.3)(0.722)^2) = -0.8784$
$x_4 = 0.4$	$y_4 = ?$	

we have AB predictor formula

$$y_4^{(P)} = y_3 + \frac{h}{24}\left[55y_3' - 59y_2' + 37y_1' - 9y_0'\right]$$

$$y_4^{(p)} = 0.722 + \frac{0.1}{24}\left[55(-0.8784) - 59(-0.9367) + 37(-0.9819) - 9(-1)\right]$$

$$y_4^{(p)} = 0.63709$$

$$y_4' = -(y_4 + x_4 y_4^2)$$

$$= -(0.63709 + (0.4)(0.63709)^2)$$

$$y_4' = -0.79944$$

Next we have $y_4^{(c)} = y_3 + h/24\,(9y_4' + 19y_3' - 5y_2' + y_1')$

$$y_4^{(c)} = 0.722 + \frac{0.1}{24}\left(9(-0.79944) + 19(-0.8784) - 5(-0.9367) + (-0.9819)\right)$$

$$y_4^{(c)} = 0.6379$$

$$y_4' = -(y_4 + x_4 y_4^2)$$

$$= -(0.6379 + (0.4)(0.6379)^2)$$

$$= -0.80066$$

apply corrector formula once again

$$y_4^{(c)} = 0.722 + \frac{0.1}{24}\left(9(-0.80066) + 19(-0.8784) - 5(-0.9367) + (-0.9819)\right)$$

$$= 0.63785$$

$$y_4^{(c)} = 0.6379$$

Thus $y(0.4) = 0.6379$

(7) Find the value of y at $x=4.4$ by applying Adams - Bashforth method given that $5x\frac{dy}{dx} + y^2 - 2 = 0$ and $y=1$ at $x=4$ initially by generating the other required values from the Taylor's polynomial.

Sol^n: we need to generate the value of y at $x = 4.1, 4.2, 4.3$

Taylor's series expansion is given by

$$y(x) = y(x_0) + (x - x_0) y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + \cdots$$

Since $x_0 = 4 \quad y_0 = 1$ by data

$$y(x) = y(4) + (x-4) y'(4) + \frac{(x-4)^2}{2} y''(4) \quad \text{— (1)}$$

Consider $5xy' + y^2 - 2 = 0$ — (2) we obtain

Substituting the initial values

$$(5)(4) y'(4) + 1^2 - 2 = 0$$

$$20y'(4) - 1 = 0$$

$$20y'(4) = 1$$

$$y'(4) = 1/20$$

$$y'(4) = 0.05$$

D. (2) w.r. to x

$$5\left[xy'' + y'\right] + 2yy' = 0$$

Substituting the initial values and $y'(4)$

$$5\left[4y''(4) + y'(4)\right] + 2y(4)y'(4) = 0$$

$$20y''(4) + 5(0.05) + 2(1)(0.05) = 0$$

$$20y''(4) + 0.25 + 0.1 = 0$$

$$20y''(4) + 0.35 = 0$$

$$20y''(4) = -0.35$$

$$y''(4) = \frac{-0.35}{20}$$

$$y''(4) = -0.0175$$

Since the value of the second derivative itself is small enough we shall approximate Taylor's series in (1) upto second degree terms only.

Substitute these values in (1)

$$y(x) = 1 + (x-4)(0.05) + \frac{(x-4)^2}{2}(-0.0175)$$

Now we need to find y at $x = 4.1$, 4.2, 4.3

(1) $y(4.1) = 1 + (4.1-4)(0.05) + \frac{(4.1-4)^2}{2}(-0.0175)$

$y(4.1) = 1.0049$

$y(4.2) = 1 + (4.2-4)(0.05) + \frac{(4.2-4)^2}{2}(-0.0175)$

$y(4.2) = 1.0097$

$y(4.3) = 1 + (4.3-4)(0.05) + \frac{(4.3-4)^2}{2}(-0.0175)$

$y(4.3) = 1.0142$ use these along with $y(4) = 1$ initially

x	y	$y' = \frac{2-y^2}{5x}$
$x_0 = 4$	$y_0 = 1$	$y_0' = \frac{2-y_0^2}{5x_0} = \frac{2-1^2}{5(4)} = 1/20 = 0.05$
$x_1 = 4.1$	$y_1 = 1.0049$	$y_1' = \frac{2-(1.0049)^2}{5(4.1)} = 0.0483$
$x_2 = 4.2$	$y_2 = 1.0097$	$y_2' = \frac{2-(1.0097)^2}{5(4.2)} = 0.04669$
$x_3 = 4.3$	$y_3 = 1.0142$	$y_3' = \frac{2-(1.0142)^2}{5(4.3)} = 0.04518$
$x_4 = 4.4$	$y_4 = ?$	

we have predictor formula

$$y_4^{(P)} = y_3 + \frac{h}{24}\left[55y_3' - 59y_2' + 37y_1' - 9y_0'\right]$$

$$= 1.0142 + \frac{0.1}{24}\left[55(0.04518) - 59(0.04669) + 37(0.0483) - 9(0.05)\right]$$

$$y_4^{(P)} = 1.01864$$

Now, $y_4' = \frac{2 - y_4^2}{5x_4}$

$$y_4' = \frac{2 - (1.01864)^2}{5(4.4)}$$

$$y_4' = 0.0437$$

Next we have

$$y_4^{(c)} = y_3 + \frac{h}{24}\left(9y_4' + 19y_3' - 5y_2' + y_1'\right)$$

$$= 1.0142 + \frac{0.1}{24}\left(9(0.0437) + 19(0.04518) - 5(0.04669) + 0.0483\right)$$

$$y_4^{(c)} = 1.0186$$

Now, $y_4' = \frac{2 - y_4^2}{5x_4}$

$$y_4' = \frac{2 - (1.0186)^2}{5(4.4)}$$

$$y_4' = 0.0437$$

again put this value in $y_4^{(c)}$

$$\therefore y_4^{(c)} = 1.0142 + \frac{0.1}{24}\left[9(0.0437) + 19(0.04518) - 5(0.04669) + 0.0483\right]$$

$$y_4^{(c)} = 1.0186$$

thus $y(4.4) = 1.0186$

⑧ Use Taylor's Series method (upto 3rd derivative term) to find y at $x = 0.1, 0.2, 0.3$ given that $\frac{dy}{dx} = x^2 + y^2$ with $y(0) = 1$. Apply milne's predictor corrector formula to find $y(0.4)$ using the generated set of initial values.

Soln:- By data

$$\frac{dy}{dx} = x^2 + y^2, \quad y_0 = 1 \quad x_0 = 0$$

Taylor's Series expansion is given by

$$y(x) = y(x_0) + (x - x_0)y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0) + \cdots$$

$$y(x) = y(0) + (x-0)y'(0) + \frac{(x-0)^2}{2}y''(0) + \frac{(x-0)^3}{6}y'''(0)$$

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2}y''(0) + \frac{x^3}{6}y'''(0) \quad \text{—①}$$

Consider $y' = x^2 + y^2$ —②

$$y'(0) = 0^2 + (y(0))^2$$

$$y'(0) = 0 + (1)^2$$

$$y'(0) = 1$$

D.② w.r.to x

$$y'' = 2x + 2yy' \quad \text{—③}$$

$$y''(0) = 2(0) + 2y(0)y'(0)$$

$$y''(0) = 0 + 2(1)(1)$$

$$y''(0) = 2$$

D

y'''

y'''

y'''

y''

D. ③ w.r. to x

$y''' = 2 + 2[y'y'' + y'y']$

$y''' = 2 + 2y'y'' + 2(y')^2$

$y'''(0) = 2 + 2y'(0)y''(0) + 2(y'(0))^2$

$y'''(0) = 2 + 2(1)(2) + 2(1)^2$

$y'''(0) = 2 + 4 + 2$

$y'''(0) = 8$

put all these in ①

$y(x) = 1 + x(1) + \frac{x^2}{2}(2) + \frac{x^3}{6}(8)$

Now, we need to find y at $x = 0.1, 0.2, 0.3$

$y(0.1) = 1 + (0.1)(1) + \frac{(0.1)^2}{\cancel{2}}(\cancel{2}) + \frac{(0.1)^3}{6} \times 8$

$y(0.1) = 1.1113$

$y(0.2) = 1 + (0.2)(1) + \frac{(0.2)^2}{\cancel{2}} \times \cancel{2} + \frac{(0.2)^3}{6} \times 8$

$y(0.2) = 1.2507$

$y(0.3) = 1 + (0.3)(1) + \frac{(0.3)^2}{2} \times 2 + \frac{(0.3)^3}{6} \times 8$

$y(0.3) = 1.426$

using these values along with $y(0) = 1$ initially, we prepare the following table

P.T.O

x	y	$y' = x^2 + y^2$
$x_0 = 0$	$y_0 = 1$	$y_0' = 0 + 1 = 1$
$x_1 = 0.1$	$y_1 = 1.1113$	$y_1' = (0.1)^2 + (1.1113)^2 = 1.2449$
$x_2 = 0.2$	$y_2 = 1.2507$	$y_2' = (0.2)^2 + (1.2507)^2 = 1.60425$
$x_3 = 0.3$	$y_3 = 1.426$	$y_3' = (0.3)^2 + (1.426)^2 = 2.12347$
$x_4 = 0.4$	$y_4 = ?$	

Consider $y_4^{(P)} = y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3')$

$$y_4^{(P)} = 1 + \frac{4(0.1)}{3}\left[2(1.2449) - (1.60425) + 2(2.12347)\right]$$

$$y_4^{(P)} = \underline{\underline{1.68433}}$$

hence $y_4' = x_4^2 + y_4^2$

$$y_4' = (0.4)^2 + (1.68433)^2$$

$$y_4' = 2.9969$$

Next we have, $y_4^{(C)} = y_2 + h/3\,(y_2' + 4y_3' + y_4')$

$$y_4^{(C)} = 1.2507 + \frac{0.1}{3}\left[1.60425 + 4(2.12347) + 2.9969\right]$$

$$= \underline{\underline{1.6872}}$$

$$y_4' = x_4^2 + y_4^2$$

$$y_4' = (0.4)^2 + (1.6872)^2$$

$$y_4' = \underline{\underline{3.0066}}$$

put y_4' again in the corrector formula

$$y_4^{(c)} = 1.2507 + \frac{0.1}{3}\left[1.60425 + 4(2.12347) + 3.0066\right]$$

$$y_4^{(c)} = 1.6875$$

$$y_4' = x_4^2 + y_4^2$$

$$y_4' = (0.4)^2 + (1.6875)^2$$

$$y_4' = 3.0076$$

put y_4' again in the corrector formula

$$y_4^{(c)} = 1.2507 + \frac{0.1}{3}\left[1.60425 + 4(2.12347) + 3.0076\right]$$

$$y_4^{(c)} = 1.6875$$

Thus $y(0.4) = 1.6875$

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